if not necessarily linearly independent, call $v_1, \dots, v_n$
a "generating"/"spanning" set

Last time: basis $v_1, \dots, v_n$ of a vector space V

- spans V
- linearly independent

Therefore, any vector $z \in V$ can be written uniquely as

$z = x_1 v_1 + \dots + x_n v_n$, where $(x_1, \dots, x_n)$ are coordinates of z

Notation for coordinates: $[z]_{\underline{v}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

where $\underline{v}$ indicates the basis $\{v_1, \dots, v_n\}$ of V

note that this notation depends on the ordered basis $\{v_1, \dots, v_n\}$, in that changing the order of $v_1, \dots, v_n$ will change the coordinates $x_1, \dots, x_n$ by a permutation

(e.g. $\binom{7}{5}$ in the ordered basis $\{e_1, e_2\}$ is $\binom{5}{7}$ in the ordered basis $\{e_2, e_1\}$)

- any two bases of V have the same $\underbrace{\text{number of elements}}_{= \dim(V)}$

- given two bases $\underline{v} = \{v_1, \dots, v_n\}$ and $\underline{w} = \{w_1, \dots, w_n\}$ of V

they are related by the change of basis/coordinate matrix

$$P_{\underline{v} \leftarrow \underline{w}} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} [w_1]_{\underline{v}}, \dots, [w_n]_{\underline{v}} \end{pmatrix}$$

where
$$w_1 = a_{11}v_1 + \dots + a_{n1}v_n$$
$$\vdots$$
$$w_n = a_{1n}v_1 + \dots + a_{nn}v_n$$

- given multiple bases of V, the change of coordinate matrices are related as in the following diagram

$$\begin{tikzcd}
v_1, \dots, v_n \arrow[d, bend left, "P_{\underline{w} \leftarrow \underline{v}}"] \arrow[dd, bend left=90, "P_{\underline{z} \leftarrow \underline{v}}"] \\
w_1, \dots, w_n \arrow[u, bend left, "P_{\underline{v} \leftarrow \underline{w}}"] \arrow[d, bend left, "P_{\underline{z} \leftarrow \underline{w}}"] \\
z_1, \dots, z_n \arrow[u, bend left, "P_{\underline{w} \leftarrow \underline{z}}"] \arrow[uu, bend left=90, "P_{\underline{v} \leftarrow \underline{z}}"]
\end{tikzcd}$$

In other words, $P_{\underline{z} \leftarrow \underline{w}} P_{\underline{w} \leftarrow \underline{v}} = P_{\underline{z} \leftarrow \underline{v}}$

$$P_{\underline{v} \leftarrow \underline{w}} = \left(P_{\underline{w} \leftarrow \underline{v}}\right)^{-1}$$

Moreover, we saw that change of coordinate matrices allow us to change coordinates from one basis to another

$$P_{\underline{w} \leftarrow \underline{v}}\left([z]_{\underline{v}}\right) = [z]_{\underline{w}}$$

- suppose $\underline{v}$ and $\underline{w}$ are two bases of $\mathbb{R}^n$.

How to find $P_{\underline{v} \leftarrow \underline{w}}$? $\quad w_i = a_{1i} v_1 + \cdots\cdots + a_{ni} v_n$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = P_{\underline{v} \leftarrow \underline{e}} P_{\underline{e} \leftarrow \underline{w}} = \left(P_{\underline{e} \leftarrow \underline{v}}\right)^{-1} P_{\underline{e} \leftarrow \underline{w}} = B^{-1} C$$

$$(v_1 \cdots\cdots v_n) = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} = P_{\underline{e} \leftarrow \underline{v}} = B$$

$\underline{e} = \{e_1, \ldots, e_n\}$.

$$(w_1 \cdots\cdots w_n) = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} = P_{\underline{e} \leftarrow \underline{w}} = C$$

TODAY:

DEF 14.1: given vector spaces V and W a function $f: W \longrightarrow V$ is called **linear** if:

- $f(w + w') = f(w) + f(w')$, $\forall w, w' \in W$
- $f(\lambda w) = \lambda f(w)$, $\forall \lambda \in \mathbb{R}, \forall w \in W$

Suppose you are given bases $\underline{v} = \{v_1, \ldots, v_m\} \subset V$
$\underline{w} = \{w_1, \ldots, w_n\} \subset W$

Let's encode any linear function $f: W \to V$ by a matrix

$$f(w_j) = a_{1j} v_1 + \dots + a_{mj} v_m \quad , \forall j \in \{1, \dots, n\}$$

$$\Downarrow$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

Gist: A determines f uniquely, given our choice of bases

$$\forall z \in W, \quad z = x_1 w_1 + \dots + x_n w_n , \quad [z]_{\underline{w}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

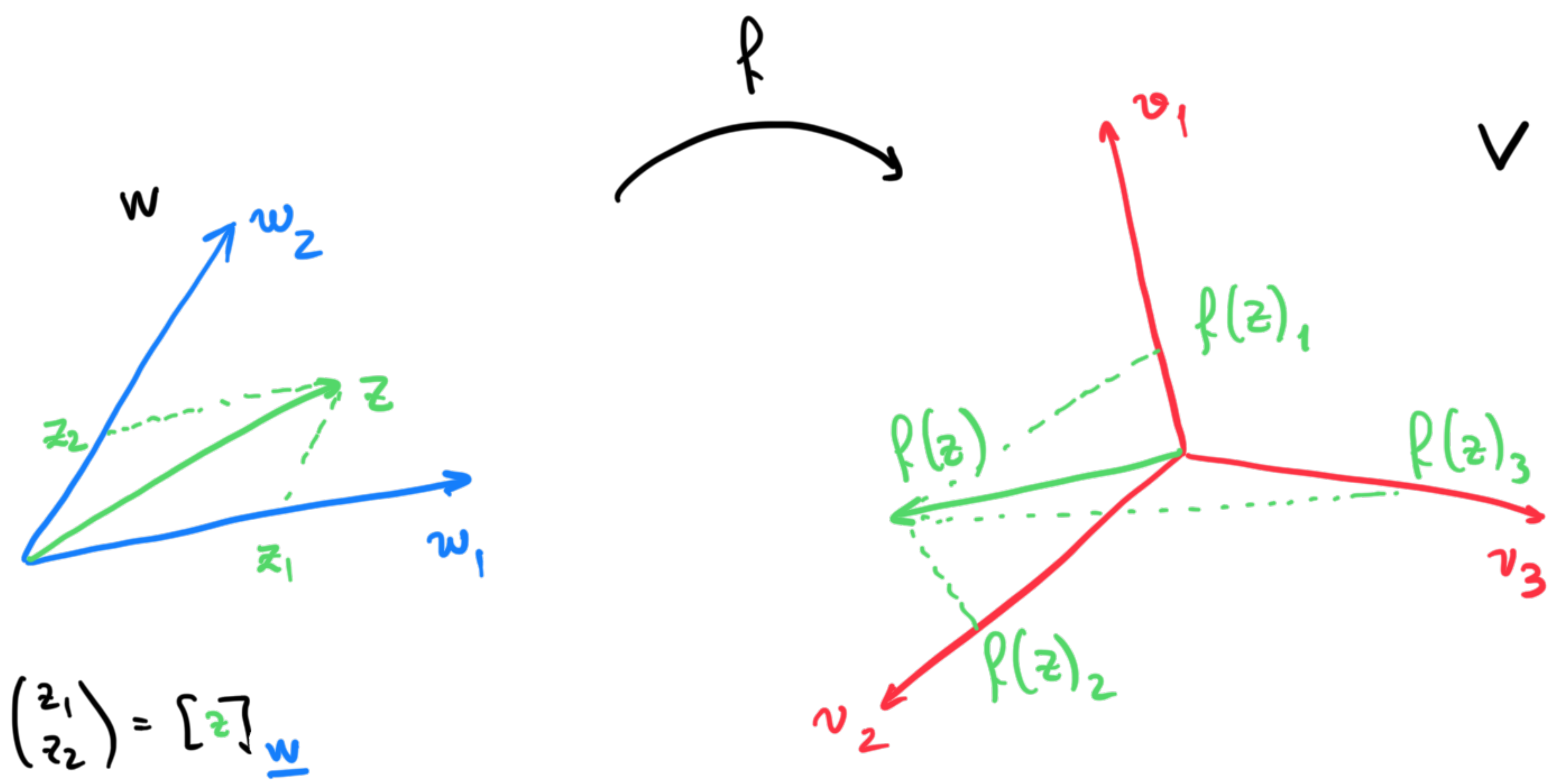
$$\Downarrow$$

$$\begin{aligned} f(z) &= f(x_1 w_1 + \dots + x_n w_n) \\ &= x_1 f(w_1) + \dots + x_n f(w_n) \\ &= x_1 (a_{11} v_1 + \dots + a_{m1} v_m) + \dots + x_n (a_{1n} v_1 + \dots + a_{mn} v_m) \\ &= \underbrace{(a_{11} x_1 + \dots + a_{1n} x_n)}_{y_1} v_1 + \dots + \underbrace{(a_{m1} x_1 + \dots + a_{mn} x_n)}_{y_m} v_m \end{aligned}$$

$$A [z]_{\underline{w}} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = [f(z)]_{\underline{v}}$$

THM 14.2: if a matrix A corresponds to a linear function $f: W \to V$ w.r.t bases $\underline{v}$ and $\underline{w}$, then

$$[f(z)]_{\underline{v}} = A\,[z]_{\underline{w}} \quad , \forall z \in W$$



- $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = [z]_{\underline{w}}$

$$\begin{pmatrix} f(z)_1 \\ f(z)_2 \\ f(z)_3 \end{pmatrix} = [f(z)]_{\underline{v}} \implies [f(z)]_{\underline{v}} = A\,[z]_{\underline{w}}$$

- $W = \mathbb{R}^n$ $\quad \underline{w}$ = standard $\quad , \quad f: \mathbb{R}^n \longrightarrow \mathbb{R}^m \rightsquigarrow A \in \mathbb{R}^{m \times n}$

  $V = \mathbb{R}^m$ $\quad \underline{v}$ = standard $\quad\quad f(x) = Ax$

- $V = W$ arbitrary $\quad \underline{w}, \underline{v}$ arbitrary, $f: W \xrightarrow{Id} V \rightsquigarrow A = P_{\underline{v} \leftarrow \underline{w}}$

fix a basis $\underline{w}$

THM 14.3: if $Z \xrightarrow{g} W \xrightarrow{f} V$

fix a basis $\underline{z}$ fix a basis $\underline{v}$

B A

then $f \circ g$ corresponds to the matrix AB
f^{-1} corresponds to the matrix A^{-1}

• given $f: W \to V$, let's construct "interesting" vector spaces.

DEF 14.4: $Ker(f) = \{ w \in W \mid f(w) = 0_V \} \subseteq W$

$$Im(f) = \{ v \in V \mid \exists\, w \in W \text{ s.t. } f(w) = v \} \subseteq V$$

$Ker(f)$ and $Im(f)$ are subspaces of W and V; Proof for $Ker(f)$ below

- $0_W \in Ker(f)$, b/c $f(0_W) = 0_V$
- $\forall\, w, w' \in Ker(f) \Rightarrow f(w) = f(w) = 0_V \Rightarrow f(w + w') = 0_V \Rightarrow w + w' \in Ker(f)$
- $\forall\, w \in Ker(f), \forall \lambda \in \mathbb{R} \Rightarrow f(w) = 0_V \Rightarrow f(\lambda w) = 0_V \Rightarrow \lambda w \in Ker(f)$

Similar, $\mathrm{Im}(f)$ is a subspace.

THM 14.5: f is injective $\iff \mathrm{Ker}(f) = \{0_W\}$

f is surjective $\iff \mathrm{Im}(f) = V$

New topic: consider the particular case of a linear function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \quad f(x) = Ax \quad \text{where } A \in \mathbb{R}^{m \times n}$$

vector space of solutions to equation $Ax=0$

$$\mathrm{Ker}(f) = \{x \in \mathbb{R}^n \mid Ax = 0\} = \mathrm{Ker}(A) = \mathrm{Null}(A)$$

"nullspace"

$$\mathrm{Im}(f) = \{b \in \mathbb{R}^m \mid Ax = b \text{ has solutions}\} = \mathrm{Col}(A)$$

"span of columns of A" = "column space of A"

Row space of A $= \mathrm{Row}(A) = \mathrm{Col}(A^T) =$ "span of rows of A"

We can use Gaussian elimination to find bases for

$\mathrm{Ker}(A)$, $\mathrm{Col}(A)$, $\mathrm{Row}(A)$

Problem 1: $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix} \rightsquigarrow \mathrm{REF}(A) = \begin{pmatrix} \boxed{1} & 0 & 1 & 1 \\ 0 & \boxed{1} & 1 & 0 \end{pmatrix}$

Problem 1: $A = \begin{pmatrix} \cdots \\ 2 & 3 & 5 & 2 \end{pmatrix}$ to REF(A) $= \begin{array}{c} \begin{pmatrix} \cdots \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \text{pivot} \quad \text{free} \end{array}$

Find a basis for $\mathrm{Ker}(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{ s.t. } A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 \right\}$

$$\mathrm{Ker}(\mathrm{REF}(A)) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{ s.t. } \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{ s.t. } \begin{array}{l} x_1 + x_3 + x_4 = 0 \\ x_2 + x_3 = 0 \end{array} \right\}$$

Denote $x_3 = s$, $x_4 = t$

$$= \left\{ \begin{pmatrix} -s-t \\ -s \\ s \\ t \end{pmatrix} \right\}_{s,t \in \mathbb{R}} = \left\{ s \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}_{s,t \in \mathbb{R}}$$

basis vectors of Ker(A)

Upshot: $\mathrm{Ker}(A) = \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \to \dim = 2$

$\quad \| $

$\#$ free columns

Problem 2: ... V spanned by $\begin{pmatrix} 1 \\ \cdots \end{pmatrix}, \begin{pmatrix} 1 \\ \cdots \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ \cdots \end{pmatrix}$

Problem 2: consider vector space V spanned by $\begin{pmatrix}1\\2\\3\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\3\\5\\2\end{pmatrix}$

Find a basis for V

- write down the matrix whose rows are your vectors

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 2 & 3 & 5 & 2 \end{pmatrix} \rightsquigarrow REF(A) = \begin{pmatrix} \boxed{1} & 0 & 1 & 1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$V = Col(A^T) \underset{\text{unchanged by Gaussian elemination}}{=\!=\!=\!=\!=} Col(REF(A)^T)$$

Pivot rows of REF(A) are linearly independent

$\Downarrow$

pivot rows form a basis

Upshot: a basis of V is given by $\begin{pmatrix}1\\0\\1\\1\end{pmatrix}$ and $\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$

$\Downarrow$

dim(V) = 2

Problem 3: V = vector space spanned by $\begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}2\\1\\3\end{pmatrix}, \begin{pmatrix}3\\2\\5\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix}$

Whittle down this generating set to a basis of V

Construct the matrix whose columns are the vectors

$A = \begin{pmatrix} 1 & 2 & 3 & 1 \end{pmatrix} \rightsquigarrow REF(A) = \begin{pmatrix} \boxed{1} & 0 & 1 & 1 \end{pmatrix}$

$$A = \begin{array}{c} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & 5 & 2 \end{pmatrix} \\ \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \end{array} \rightsquigarrow REF(A) = \begin{array}{c} \begin{pmatrix} 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \end{array} \end{array}$$

$$V = \mathrm{Col}(A) \neq \mathrm{Col}(REF(A))$$

but any linear relation between the columns of A is reflected by linear relation between the columns of $REF(A)$

specifically, c_1 and c_2 are independent, but $c_3 = c_1 + c_2$, $c_4 = c_1$

$\Downarrow$

b_1 and b_2 are independent, but $b_3 = b_1 + b_2$, $b_4 = b_1$

$\Downarrow$

a basis of V is given by $b_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ $b_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

A basis for $\mathrm{Col}(A)$ is given by the pivot columns of A ($\neq$ pivot columns of $REF(A)$)

$\Downarrow$

$\dim(V) = 2$

New topic: Vector spaces of polynomials

$\downarrow$

$$\{ a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_d x^d \}$$

$\mathbb{R} \quad \mathbb{R} \quad \mathbb{R} \qquad \mathbb{R}\setminus 0$

a degree d polynomial

Ex: 2 is a polynomial of deg 0

$3+5x$ —— // —— 1

$7-x^3$ —— // —— 3

Polynomials form a vector space (w.r.t. addition)

basis $\{x^0, x^1, x^2, x^3, x^4, \ldots\ldots\}$ monomials

$$1+5x-9x^3 = 1\cdot x^0 + 5\cdot x^1 + 0\cdot x^2 + (-9)\cdot x^3$$

DEF 14.6: $\mathbb{P}_d = \{$vector space of polynomials of deg $\leq d\}$

$$\underbrace{(1+5x-7x^2)}_{\text{deg } 2} + \underbrace{(9-17x+7x^2)}_{\text{deg } 2}$$

$$\| $$

$$\underbrace{10-12x}_{\text{deg } 1}$$

a basis of $\mathbb{P}_d$ is given by $\{x^0, x^1, x^2, \ldots, x^d\}$

$$\dim(\mathbb{P}_d) = d+1$$

- $f: \mathbb{P}_d \longrightarrow \mathbb{P}_d$ derivative, i.e. $f(x^k) = (x^k)' := kx^{k-1}$

Claim: f is a linear function on $\mathbb{P}_d$ $\iff$ $(P+Q)' = P' + Q'$

$(\lambda P)' = \lambda \cdot P'$